

Overlap^{*}

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The overlap formulation of regulated vectorial and chiral gauge theories is reviewed. Ostensibly new constructions, based on the Ginsparg-Wilson relation are essentially just overlap with new notation. At present there exists no satisfactory realization of chiral symmetries outside perturbation theory which is structurally different from the overlap.

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I. Introduction

My talk is divided into five parts. In the first part the relevance of the chiral fermion issue to fundamental particle physics and to numerical QCD will be explained. In the second part the basic ideas of the overlap construction will be reviewed. In the third part I shall focus on the algebraic (as opposed to kinematic) meaning of the overlap. In the fourth part I shall discuss in greater detail the overlap Dirac operator on the lattice. In the last part I shall deal with anomalies, the associated Berry phases and Förster-Nielsen-Ninomiya gauge averaging in the context of naturalness. My main partner in the overlap construction was R. Narayanan. I have also collaborated with P. Huet, Y. Kikukawa, A. Yamada and P. Vranas. Important contributions to the overlap development were also made by S. Randjbar-Daemi and J. Strathdee.

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II. Relevance

Fermion chirality is one of the most fundamental properties of Nature where it appears in association with gauge interactions. This combination can decouple from a more fundamental theory if anomalies cancel. It is however not trivial to achieve this decoupling outside perturbation theory, and for a while it was even thought an impossible feat. The overlap provides an example where the decoupling works in a non-perturbative framework. In a unitary framework the mechanism requires an infinite number of fermion degrees of freedom per unit four space-time volume. It would be a major achievement for lattice field theory if it turned out that a similar mechanism is operative in Nature. More details regarding general relevance can be found in [1].

III. Basic structure

The overlap is an outgrowth of a long sequence of papers starting from the fundamental discoveries in [2], [3], [4] and more specifically [5], [6], [7], [8]. My first papers on the subject were all written with Narayanan who stayed my collaborator for a long time [9], [10, 11, 12]. In parallel, important contributions to the overlap were made in [13]. Later, I continued to simplify the overlap in the vector-like context [14].

Let us start with a formally vector-like gauge theory:

$$\mathcal{L}_\psi = \bar{\psi} \not{D} \psi + \bar{\psi} (P_L \mathcal{M} + P_R \mathcal{M}^\dagger) \psi. \quad (1)$$

$\bar{\psi}$ and ψ are Dirac fermions and the mass matrix \mathcal{M} is infinite. It has a single zero mode but its adjoint has no zero modes. As long as $\mathcal{M} \mathcal{M}^\dagger > 0$ this setup is stable under small deformations of the mass matrix implying that radiative corrections will not wash the zero mode away.

Kaplan's domain wall suggests the following realization:

$$\mathcal{M} = -\partial_s - f(s), \quad (2)$$

where $s \in (-\infty, \infty)$ and f is fixed at $-\Lambda'$ for negative s and at Λ for positive s ($\Lambda', \Lambda > 0$). There is no mathematical difficulty associated with the discontinuity at $s = 0$.

The infinite path integral over the fermions is easily “done”: on the positive and negative segments of the real line respectively one has propagation with an s -independent “Hamiltonian”. The infinite extent means that at $s = 0$ the path integrals produce the overlap (inner product) between the two ground states of the many fermion systems corresponding to each side of the origin in s . The infinite extent also means infinite exponents linearly proportional to the respective energies - these factors are subtracted. One is left

with the overlap formula which expresses the chiral determinant as $\langle v'\{U\}|v\{U\}\rangle$. The states are in second quantized formalism. By convention, they are normalized, but their phases are left arbitrary. This ambiguity is essential, as we shall see later on. It has no effect in the vector-like case. In first quantized formalism the overlap is:

$$\langle v'\{U\}|v\{U\}\rangle = \det_{k'k} M_{k'k}. \quad (3)$$

The elements of the matrix M are the overlaps between single body wave-functions, $M_{k'k} = v_{k'}^\dagger v_k$. The v' 's span the negative energy subspace of $H' \sim \gamma_5(\not{D}_4 + \Lambda')$ and the v 's span the negative energy subspace of $H \sim \gamma_5(\not{D}_4 - \Lambda)$. I used continuum like notation to emphasize that the Hamiltonians are arbitrary regularizations of massive four dimensional Dirac operators with large masses of opposite signs.

The Hamiltonians only enter as defining the Dirac seas and there is no distinction between the different levels within each sea; all that matters is whether a certain single particle state has negative or positive energy. Thus, all the required information is also contained in the operators $\epsilon = \varepsilon(H)$ and $\epsilon' = \varepsilon(H')$ where ε is the sign function. Thus the v' 's are all the -1 eigenstates of ϵ' and the v 's are all the -1 eigenstates of ϵ . To switch chiralities one only has to switch the sign of the Hamiltonians. This is a result of charge conjugation combined with a particle-hole transformation.

When Λ' is taken to infinity in lattice units one is left with $\epsilon' = \gamma_5$ with no gauge field dependence. On the other hand, ϵ always maintains a dependence on the gauge background and its trace gives the topological charge of the gauge field.

More recently, a construction starting from the GW relation [15] has been presented as new [16]. This is misleading, since the “new” construction merely reproduces the structure of the overlap. The only differences are in technicalities surrounding the phase choice of the overlap. These technicalities are not trivial, but I would be surprised if any new particle physics were produced by this effort. To be sure, a successful completion of the phase choice program would be news in mathematical physics, and would make me happy. The proposal of the “new” construction was preceded by the observation that the overlap satisfies the GW relation [17], [18] and that the overlap-GW relation is essentially one to one [19], [20]. Thus, it should have been obvious that starting from the GW relation will lead to the overlap structure in the chiral case. The insistence of the proponents of the “new” construction to obscure its structural equivalence to the overlap has generated much confusion and waste of time, deflecting energy from some valid physics problems that are still open. For example, it is an important open question whether a genuinely different natural construction of chiral gauge theories is possible.

IV. Algebraic meaning

The GW relation is a complicated way to write down what is best described as the algebraic content of the overlap. It is much simpler to rephrase and slightly generalize: The setup is that of a “Kato pair” [21], an arrangement further developed in the Quantum Hall Effect context by Avron, Seiler and Simon [22] and others [23].

The core objects are the two hermitian reflections, ϵ and ϵ' introduced above. They are equivalent to two orthogonal projectors P, Q , the Kato pair ($\epsilon = 1 - 2Q$, $\epsilon' = 1 - 2P$). Defining

$$h = \frac{1}{2}(\epsilon + \epsilon'), \quad s = \frac{1}{2}(\epsilon - \epsilon'), \quad (4)$$

we arrive at the following fundamental relations:

$$h^2 + s^2 = 1, \quad \{h, s\} = 0. \quad (5)$$

These relations allow us to write any element in the algebra generated by h and s as $f_1(h) + sf_2(h^2) + hsf_3(h^2)$, where h^2 is a central element. Geometrically, P and Q generate two orthogonal decompositions of the entire space \mathcal{V} . A natural operator mapping one subspace of one decomposition to another subspace of the other decomposition was introduced by Kato. It coincides with overlap Dirac operator.

$$D_o = \frac{1}{2}(1 + \epsilon'\epsilon) = \epsilon'h = h\epsilon. \quad (6)$$

That it connects the two decompositions is evident from $PD_o = D_oQ$. In the overlap context D_o was found via the following easily proven identity:

$$\det D_o = |\det M|^2. \quad (7)$$

The following decomposition of the Hilbert space into subspaces invariant under h and s holds:

$$\mathcal{V} = \text{Ker}(h) \oplus \text{Ker}(s) \oplus \mathcal{V}_\perp. \quad (8)$$

$\text{Tr}h|_{\mathcal{V}_\perp} = \text{Tr}s|_{\mathcal{V}_\perp} = 0$ by virtue of $\{h, s\} = 0$. One also has

$$\text{Ker}(h) = \text{Ker}(s - 1) \oplus \text{Ker}(s + 1), \quad \text{Ker}(s) = \text{Ker}(h - 1) \oplus \text{Ker}(h + 1). \quad (9)$$

A special property of the Kato pair in the overlap context is that h, s are both gauge field dependent and that for all backgrounds $\text{Tr}h = \text{Tr}s$. One can introduce now a definition of index

$$\text{index}(\epsilon', \epsilon) = \dim \text{Ker}(s - 1) - \dim \text{Ker}(s + 1) = \text{Tr}s^{2n+1} \quad (10)$$

for any integer $n \geq 0$. To prove n -independence of the right hand side use $s^2 = 1 - h^2$ and $Trh^{2k}s = Trh^{2k-1}sh = -Trh^{2k}s$. Reversing the roles of h and s switches the sign of the index. It turns out that the index is exactly the topological charge of the gauge field as sensed by the overlap [24]. Moreover, if $s = 1$ and $s = -1$ are isolated points of the spectrum of s and both $Ker(s + 1)$ and $Ker(s - 1)$ are finite dimensional the definition extends to $dim\mathcal{V} = \infty$.

Until now the roles of h and s were symmetrical, and there is no massless fermion in sight. The entire setup is pure mathematics. The kinematic part of the construction is in the relationship between the reflections (or projectors) and the massive Dirac operator. This is the basis of the lattice realization of the overlap idea to be covered in the next section. Here, we shall devote a few lines to explain why the Kato pair is an algebraic set-up which is essentially equivalent to the Ginsparg Wilson relation. The Ginsparg Wilson relation is:

$$\{D, \gamma_5\} = 2D\gamma_5RD. \quad (11)$$

One adds to the above the following requirements: (1) γ_5 -hermiticity, $\gamma_5 D \gamma_5 = D^\dagger$; (2) $[R, \gamma_\mu] = 0$; (3) $R^\dagger = R > 0$; R is local in the site index. It goes without saying that D and R transform covariantly under gauge transformations and that D should be local in some sense.

Choosing $\epsilon' = \gamma_5$ it is obvious that D_o satisfies the GW relation and the extra requirements with $R = 1$. ϵ , being associated with very massive fermions, will be essentially local. In the lattice context, as we shall see in the next section, some gauge backgrounds need to be excluded if locality is to hold absolutely.

What may be less obvious is the other direction of the equivalence, namely, that the GW relation implies a Kato pair. Starting from the GW relation takes one back to the overlap situation at the algebraic level, i.e., before the explicit form of the reflection ϵ is given. The reflection ϵ' , which is quite free in the overlap, is restricted in the GW case to γ_5 .

The first step is to eliminate R , by showing that any solution with $R = 1$ produces a solution with an arbitrary R . This can be done in at least two ways [25]. The first starts from the main observation that the basic GW relation [20] can be written as

$$\{D^{-1}(R) - R, \gamma_5\} = 0 \quad (12)$$

and therefore setting $D^{-1}(R) - R = D^{-1}(1) - 1$ ensures that $D(R)$ is a solution if $D(1)$ is. Although the inverse of $D_\chi \equiv \frac{1}{D^{-1}(R) - R}$ is non-local by the well known no-go theorems, D_χ^{-1} has only zeros at the ‘‘doubler’’ locations in the free case, and R removes these zeros from $D^{-1}(R)$, so that $D(R)$ ends up local. Once $D(R)$ is local and doubler free in the

free case, locality in the presence of lattice backgrounds that are sufficiently close to the continuum is assured. The second way is to set $D(R) = \frac{1}{\sqrt{R}}D(1)\frac{1}{\sqrt{R}}$. So, nothing is lost or gained in principle by fixing $R = 1$.

For $R = 1$ it is trivial to go back to the reflections: one just looks at the formula for D_o to define the operator $\gamma_5[2D(1) - 1]$. It is trivial that this operator is hermitian and squares to unity by the basic GW formula [19]. So, $D(1)$ is just another D_o . Now it is trivial to extract h and s , since ϵ' is given as γ_5 . Also, it is a trivial matter to go to the matrix M and the chiral case.

V. Concrete realization

The massless fermions enter as a representation of the difference between Dirac operators with masses of opposite sign. There are no particular difficulties to regularize *massive* Dirac fermions, in the continuum or on the lattice. On the lattice the spectra of the Dirac operators get compactified and the regularization is fully non-perturbative.

First, some notation: Let $U_\mu(x)$ denote $SU(n)$ link matrices on a finite d -dimensional hypercubic lattice. The directional parallel transporters T_μ are:

$$T_\mu(\psi)(x) = U_\mu(x)\psi(x + \hat{\mu}). \quad (13)$$

Let $G(g)$ with $(G(g)\psi)(x) = g(x)\psi(x)$ describe a gauge transformation.

$$G(g)T_\mu(U)G^\dagger(g) = T_\mu(U^g), \quad \text{where, } U_\mu^g(x) = g(x)U_\mu(x)g^\dagger(x + \hat{\mu}). \quad (14)$$

A lattice replacement of the massive continuum Dirac operator, $D(m)$, is an element in the algebra generated by T_μ , T_μ^\dagger , γ_μ . Thus, $D(m)$ is gauge covariant. The Wilson Dirac operator, $D_W(m)$ can be written as:

$$D_W = m + \sum_\mu (1 - V_\mu); \quad V_\mu^\dagger V_\mu = 1; \quad V_\mu = \frac{1 - \gamma_\mu}{2}T_\mu + \frac{1 + \gamma_\mu}{2}T_\mu^\dagger. \quad (15)$$

The hermitian Wilson Dirac operator is $H_W(m) = \gamma_{d+1}D_W(m)$. In terms of the continuum covariant derivative D_μ^c , for lattice spacing a , $V_\mu = e^{-a\gamma_\mu D_\mu^c}$, with no sum on μ .

We choose the following norm definition for matrices A : $\|A\| = \sqrt{\lambda_{\max}(A^\dagger A)}$. The norm of a gauge covariant matrix is gauge invariant. λ_{\max} is a maximal eigenvalue.

Our two reflections are defined by:

$$\epsilon' = \lim_{m \rightarrow +\infty} \varepsilon(H_W(m)) = \gamma_{d+1}; \quad \epsilon = \varepsilon(H_W(m)) \quad \text{with} \quad -2 < m < 0. \quad (16)$$

For ϵ to be defined everywhere we need $H_W^2(m) > 0$. This can be assured if the pure gauge action enforces an upper bound on the norms of $[T_\mu, T_\nu]$ [26]. The following inequality, when meaningful, specifies what constraint would be sufficient:

$$\left[\lambda_{\min}(H_W^2(m))\right]^{\frac{1}{2}} \geq \left[1 - (2 + \sqrt{2}) \sum_{\mu > \nu} \|[T_\mu, T_\nu]\|\right]^{\frac{1}{2}} - |1 + m|, \quad -2 < m < 0. \quad (17)$$

The upper bound on $\|[T_\mu, T_\nu]\| = \|1 - P_{\mu\nu}\|$ is compatible with the continuum limit, where the T_μ 's almost commute: $[T_\mu, T_\nu]^\dagger [T_\mu, T_\nu] = (1 - P_{\mu\nu})^\dagger (1 - P_{\mu\nu})$, $P_{\mu\nu} = T_\nu^\dagger T_\mu^\dagger T_\nu T_\mu$, $(P_{\mu\nu}\psi)(x) = U_{\mu\nu}(x)\psi(x)$, $U_{\mu\nu}(x) = U_\nu^\dagger(x)U_\mu^\dagger(x + \hat{\nu})U_\nu(x + \hat{\mu})U_\mu(x)$. Any pure gauge action with the right continuum limit and close to it will strongly prefer configurations where all $U_{\mu\nu}(x)$ are close to unit matrix.

On the other hand, a latticized instanton can be smoothly deformed to a trivial configuration, so there exist gauge field backgrounds for which $H_W^2\{U\}(m)$ has an exact zero mode. This is true of any m in the range $(-2, 0)$. The above analysis tells us that at least one plaquette will be relatively big for this configuration. Unless we enforce the constraint fully, there always will be possibly rare configurations where ϵ is not well defined.

To fully appreciate the differentiation between the algebraic character of the GW relation and the more complete framework of the overlap, let me give some examples of bad solutions to the GW relation:

$$D_{GW}^{(1)} = \frac{1}{2}(1 + \gamma_5 \varepsilon(H_W(m_1))), \quad D_{GW}^{(2)} = \frac{1}{2}(1 - \gamma_5 \varepsilon(H_W(m_2))); \quad m_{1,2} > 0. \quad (18)$$

$$D_{GW}^{(3)} = \frac{1}{2}(1 - \gamma_5 \varepsilon(H_W(m_3))); \quad -2 < m_3 < 0. \quad (19)$$

Solutions (1) and (2) have always trivial index, (1) has no massless fermions at all, while (2) has 16. (3) is motivated by the h, s symmetry, describes 15 fermions and has a nontrivial index, but 7 fermions contribute to it one way, and 8 the other.

VI. Anomalies

The absolute value of $\langle v' | v \{U\} \rangle$ is well defined and gauge invariant, but the complex number itself is defined only up to phase. In the general case we have a $U(1)$ bundle defined over the space of gauge fields with $H_W^2\{U\}(m) > 0$, a gauge invariant condition. $|v'\rangle$ is gauge field independent and gauge invariant and $|v\{U\}\rangle$ comes from a gauge covariant Hamiltonian. Thus, the $U(1)$ bundle can be viewed as defined over the space of gauge orbits. With the exclusion of the points $\det H_W\{U\}(m) = 0$ this space can support non-trivial $U(1)$ bundles as required for anomalies.

A more familiar way for physicists is to think in terms of Berry phases. The main point is that the matrices $H_W\{U\}$ are smooth in the link variables and this smoothness is inherited by the states $|v\{U\}\rangle$. With it come the inevitable Berry phase factors. The Berry phase factors come from parallel transport on the space of gauge fields with Berry's connection \mathcal{A} . Assuming some choice of states $|v\{U\}\rangle$ (possibly requiring patches with overlays) Berry's connection is the following one-form over each patch in the space of gauge fields:

$$\mathcal{A} = \langle v\{U\} | dv\{U\} \rangle. \quad (20)$$

Berry's phases come from Wilson loop factors associated with \mathcal{A} . This makes it obvious why they are independent of the choice of phases for the $|v\{U\}\rangle$. The non-triviality of Berry's phase is locally measured by the associated abelian field strength, a globally defined two-form that usually does not vanish:

$$\mathcal{F} = d\mathcal{A}. \quad (21)$$

While \mathcal{A} depends on the phase choices of the $|v\{U\}\rangle$, \mathcal{F} does not, a fact which is reflected by expressing \mathcal{F} in terms of the reflection $\epsilon = \varepsilon(H_W(m))$ alone:

$$\mathcal{F} = -\frac{1}{4} \text{Tr}[\varepsilon(H_W(m)) \, d\varepsilon(H_W(m)) \, d\varepsilon(H_W(m))]. \quad (22)$$

To understand the meaning of \mathcal{A} let us review briefly one way of looking at anomalies in the continuum. We are restricting our attention to the non-abelian case. After all, in the four dimensional abelian case the gauge coupling isn't asymptotically free and the entire issue of mathematical existence of an interacting chiral gauge theory is a moot point.

Once one has some definition of a regulated chiral determinant one can take a variation with respect to the vector potential $A_\mu(x)$ to get a current, a nonlocal functional $J_{\text{consistent}}^\mu(A)$ of the gauge field $A_\mu(x)$. If the regulated chiral determinant is gauge invariant $J_{\text{consistent}}^\mu(A)$ transforms covariantly under gauge transformations, but when there are anomalies and the regulated chiral determinant is not gauge invariant $J_{\text{consistent}}^\mu(A)$ has a complicated transformation. However, one can always define a covariant current, $J_{\text{covariant}}^\mu(A) = J_{\text{consistent}}^\mu(A) - \Delta J^\mu(A)$ which does transform covariantly. Moreover, $\Delta J^\mu(A)$ is a local, calculable functional of $A_\mu(x)$ [27]. One cannot use $J_{\text{covariant}}^\mu(A)$ to reconstruct a gauge invariant regulated determinant because it cannot be written as the variation of a functional of $A_\mu(x)$ since it does not obey the Wess-Zumino consistency conditions. If anomalies do cancel one can find a regularization (at least in perturbation theory) with $\Delta J^\mu(A) = 0$; then $J_{\text{consistent}}^\mu(A) = J_{\text{covariant}}^\mu(A)$ can be regulated first and a regulated gauge invariant determinant can be constructed.

The main observation is that Berry's connection \mathcal{A} on the lattice (when summed over all fermion multiplet contributions) plays the role of $\Delta J^\mu(A)$ [28]. To see this we view the lattice currents as one forms over the space of gauge fields.

$$J_{\text{consistent}} = \frac{d\langle v'|v\rangle}{\langle v'|v\rangle} = \frac{\langle v'|dv\rangle_\perp}{\langle v'|v\rangle} + \mathcal{A}. \quad (23)$$

By definition, $\langle v|dv\rangle_\perp = 0$. The covariant current is

$$J_{\text{covariant}} = \frac{\langle v'|dv\rangle_\perp}{\langle v'|v\rangle}. \quad (24)$$

and it is quite obvious that it has to transform covariantly because it is unaffected by a phase change in the choice of the $|v\{U\}\rangle$ and hence is insulated of any gauge breaking step.

\mathcal{A} only depends on the overlap of the state $|v\{U\}\rangle$ with a state $|v\{U'\}\rangle$ with U' close to U . This is in contrast to the overlap or the currents where $|v\{U\}\rangle$ is overlapped with a very different state, $|v'\rangle$. This is why \mathcal{A} is local in the gauge fields but the currents are not [28], [29].

This interpretation of \mathcal{A} is made solid by the following result: Consider an anomalous theory. One can find a two parameter family of gauge backgrounds which becomes a two torus when restricted to orbit space. \mathcal{F} can also be restricted to orbit space because it is explicitly gauge invariant. One can integrate \mathcal{F} over this two torus and the result is proportional to an integer. The integer is precisely the same as the one entering the anomaly coefficient. The result can be easily shown to be consistent with a known continuum expression. On the lattice this result means that there are “monopole” like singularities in \mathcal{A} . These singularities cannot be removed by small deformations of the matrix $H_W\{U\}(m)$. Thus, we cannot envisage altering $H_W\{U\}(m)$ so that \mathcal{A} vanish. But, if anomalies do cancel the obstruction is removed.

On the basis of this I propose two conjectures [28]:

- Iff anomalies cancel can one deform the total Hamiltonian so that one attains $\mathcal{F} = 0$ without the overlap going through any singularity.
- If $\mathcal{F} = 0$ there is a natural phase choice for $|v\{U\}\rangle$, determined by parallel transport with respect to an \mathcal{A} , such that the action of gauge transformations is non-projective: For any gauge transformation $g(x)$, its representation on the fermions, $G(g)$ obeys

$$G(g)|v\{U\}\rangle = |v\{U^g\}\rangle. \quad (25)$$

If these conjectures are true it would follow that one could fine tune $H_W\{U\}(m)$ so that, provided anomalies cancel, a natural smooth gauge invariant phase choice for the overlap exists.

Still, this solution, if it indeed works, looks somewhat contrived for Nature because it requires a fine tuning of the Hamiltonian used in the construction. I believe that this fine tuning is not necessary. If a fine tuned Hamiltonian exists, any Hamiltonian close enough to it would also work, although a residual gauge dependence would be present. One would simply average over the gauge degrees of freedom (integrate them out) and this would have no effect on the continuum limit [30]. If anomalies do not cancel, the gauge dependence is always large enough that gauge averaging induces new non-local terms in the action. Indeed, the gauge dependence contains a lattice version of a nontrivial Wess-Zumino term. When anomalies cancel but one still has a small amount of gauge breaking, gauge averaging only adds some extra gauge invariant local terms to the action. This has been shown numerically in a two dimensional chiral model [12].

There exists an important special case where there are anomalies but \mathcal{F} vanishes. It is in four dimensions with gauge group $SU(2)$ and Weyl fermions in the $I = \frac{1}{2}, \frac{5}{2}, \dots$ representation. Then the matrix H_W can be made real by a global base choice. Still, Berry's phase factors can take the values ± 1 and thus Witten's anomaly is reproduced [31]. This again shows that in the overlap all anomalies are encoded in Berry's phases.

Although the overlap at present does not provide a rigorous construction for chiral gauge theories in the non-abelian case, it has passed the boundary of physical plausibility, relegating the completion of this construction to the branch of mathematical physics. For Physics the main question is: Does there exist a truly new way to regulate chiral symmetries, or is the way opened by Kaplan and by Frolov and Slavnov and subsequently realized by the overlap unique in some sense ?

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References

- [1] H. Neuberger, hep-lat/9909042.
- [2] S. Adler, *Phys. Rev. D* **177**, 2426 (1969); J. S. Bell, R. Jackiw, *Nuovo Cimento* **60A**,

- 47 (1969); S. Adler, W. Bardeen, *Phys. Rev. D* **182**, 157 (1969).
- [3] B. Zumino, “Chiral Anomalies and Differential Geometry” Les Houches 1983, Proceedings, Relativity, Groups and Topology, (QC6:G7:1983); R. Stora, “Continuum Gauge Theories”, Cargese Summer Institute 1976, (QC174.45:C35:1976).
 - [4] R. D. Ball, *Phys. Rept.* **182**, 1 (1989); L. Alvarez-Gaume, “An Introduction to Anomalies”, Erice School Math.Phys.1985:0093 (QCD161:I569:1985).
 - [5] C. Callan, J. Harvey, *Nucl. Phys. B* **250**, 427 (1984).
 - [6] D. Boyanowski, E. Dagotto, E. Fradkin, *Nucl. Phys. B* **285**, 340 (1987).
 - [7] D. B. Kaplan, *Phys. Lett. B* **288**, 342 (1992).
 - [8] S. A. Frolov, A. A. Slavnov, *Phys. Lett. B* **309**, 344 (1993).
 - [9] R. Narayanan, H. Neuberger, *Phys. Lett. B* **302**, 62 (1993).
 - [10] R. Narayanan, H. Neuberger, *Nucl. Phys. B* **412**, 574 (1994).
 - [11] R. Narayanan, H. Neuberger, *Nucl. Phys. B* **443**, 305 (1995).
 - [12] R. Narayanan, H. Neuberger, *Phys. Lett. B* **393**, 360 (1997); *Nucl. Phys. B* **477**, 521 (1996); *Phys. Lett. B* **348**, 549 (1995). Y. Kikukawa, R. Narayanan, H. Neuberger, *Phys. Rev. D* **57**, 1233 (1998); *Phys. Lett. B* **105**, 399 (1997).
 - [13] S. Randjbar-Daemi, J. Strathdee, *Phys. Lett. B* **348**, 543 (1995), *Nucl. Phys. B* **443**, 386 (1995); *Phys. Lett. B* **402**, 134 (1997); *Nucl. Phys. B* **461**, 305 (1996); *Nucl. Phys. B* **466**, 335 (1996).
 - [14] H. Neuberger, *Phys. Lett. B* **417**, 141 (1998).
 - [15] P. Ginsparg, K. Wilson, *Phys. Rev. D* **25**, 2649 (1982).
 - [16] M. Lüscher, hep-lat/9909150.
 - [17] H. Neuberger, *Phys. Rev. D* **57**, 5417 (1998).
 - [18] H. Neuberger, *Phys. Lett. B* **427**, 353 (1998).
 - [19] R. Narayanan, *Phys. Rev. D* **58**, 097501 (1998).
 - [20] T. W. Chiu, S. V. Zenkin *Phys. Rev. D* **59**, 074501 (1999).

- [21] T. Kato, “Perturbation Theory for Linear Operators”, Springer-Verlag, Berlin, 1984.
- [22] J. Avron, R. Seiler, B. Simon, *Commun. Math. Phys.* **159**, 399 (1994).
- [23] S. Borac, *J. Math. Phys.* **36**, 863 (1995)
- [24] R. Narayanan, H. Neuberger, *Phys. Rev. Lett.* **71**, 3251 (1993).
- [25] T. W. Chiu, hep-lat/9910029.
- [26] H. Neuberger, hep-lat/9911004.
- [27] W. Bardeen, B. Zumino, *Nucl. Phys. B* **244**, 421 (1984)
- [28] H. Neuberger, *Phys. Rev. D* **59**, 085006 (1999).
- [29] S. Randjbar-Daemi, J. Strathdee, *Phys. Lett. B* **402**, 134 (1997).
- [30] D. Förster, H. B. Nielsen, M. Ninomiya, *Phys. Lett. B* **94**, 135 (1980).
- [31] H. Neuberger, *Phys. Lett. B* **437**, 117 (1998); *Phys. Lett. B* **434**, 99 (1998).